The Positive Fragments of *Grundgesetze*

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1. Introduction

Frege Theorem is a cornerstone of Neo-Fregeanism, the contemporary heir to Frege's Logicism. *Frege Theorem* shows that Peano Arithmetic can be proven, in second-order logic, from Hume's Principle. Hume's Principle is that the number of *F*s is the same as the number of *G*s if, and only if, the *F*s and the *G*s are equinumerous. Formally:

$$\#F = \#G \leftrightarrow G \approx F$$

where $G \approx F$ is the abbreviation of the following sentence:

$$\exists R \forall x(Fx \rightarrow \exists y(Gy \wedge \forall z(Rxz \leftrightarrow z = y))) \wedge (Gx \rightarrow \exists y(Fy \wedge \forall z(Rzx \leftrightarrow z = y)))$$

Generally, the theory, consisting of standard second-order theory and Hume's Principle, is called *Frege Arithmetic*. However, *Frege Theorem* doesn't amount to the establishment of Frege's Logicism, because neither Hume's Principle is considered as analytical truth, nor is second-order logic as logic in strict sense. Most challenges to second-order logic are related with its comprehension axiom, which asserts that every formula defines a concept or relation. Formally:

$$\exists X \forall x(Xx \leftrightarrow \varphi(x)) \quad \textit{where } X \textit{ does not occur free in } \varphi(x)$$

It is generally argued that comprehension axiom claims too much and involves some kind of circularity, i. e. defining a concept by quantifying over all the concepts.

In *Frege: Philosophy of mathematics*, Dummett analyzes Frege's paradox, which results from comprehension axiom and Basic Law V. Basic Law V is that the extension of the concept *F* is the same as the extension of the concept *G* if, and only if, every *F* is a *G* and every *G* is an *F*, i.e. *F* and *G* are equivalent. Generally, the theory, consisting of standard second-order logic and Basic Law V, is called *Grundgesetze*. Dummett ascribes the paradox to impredicativity, i.e. the circularity involved in comprehension axiom. (Dummett 1991, 217-222) Motivated by Dummett's observation, Heck gives a model-theoretic proof, which shows that the schematic version of Basic Law V is consistent not only with simple predicative comprehension, but also with ramified predicative comprehension. Simple predicative comprehension is:

$$\exists X \forall x(Xx \leftrightarrow \varphi(x))$$

where $\varphi(x)$ *contains no bounded second-order variables*

With second-order variables subscripted, ramified predicative comprehension is:

$\exists X_n \forall x(X_n x \leftrightarrow \varphi(x))$

where $\varphi(x)$ *contains neither bounded variables of types greater than or equal to n, nor free variables of type greater than n*

The schematic version of Basic Law V is:

$$\varepsilon[\lambda x\ \varphi(x)] = \varepsilon[\lambda x\ \psi(x)] \leftrightarrow \forall x(\varphi(x) \leftrightarrow \psi(x))$$

Generally, predicative comprehension is also called Π^1_0-comprehension, i.e. formulas defining concepts are Π^1_0-formulas. A formula is called Π^1_0 if it contains no second-order quantifiers. Heck shows that the relativizations of the axioms of Robinson arithmetic (Q) to Num(x) can be proven from Hume's Principle within simple predicative second-order logic, i.e. Q is interpretable in simple predicative fragment of *Frege Arithmetic* (Heck 1996). Num(x) is defined as following:

$$\text{Num}(x) \leftrightarrow \exists F(x = \#F)$$

Heck also shows that, relative to Frege's definitions of zero, predecession and natural number, Q is interpretable in ramified *Frege Arithmetic* (Heck 2010).

Aside from Heck's proof, Burgess gives a proof-theoretic proof, which shows that the axiomatic version of Basic Law V is consistent with predicative comprehension. The axiomatic version of Basic Law V is:

$$\varepsilon X = \varepsilon Y \leftrightarrow \forall z(Xz \leftrightarrow Yz)$$

The theory, consisting of predicative comprehension and the axiomatic version of Basic Law V, is called PV or P^1V. With the second-order variable superscripted, the axioms of PV can be changed into:

$$\exists X^0 \forall x(X^0 x \leftrightarrow \varphi(x))$$

where $\varphi(x)$ *contains neither* X *nor bounded second-order variables*

$$\varepsilon X^0 = \varepsilon Y^0 \leftrightarrow \forall z(X^0 z \leftrightarrow Y^0 z)$$

P^nV is obtained by adding to $P^{n-1}V$ the following axioms:

$$\exists X^{n-1} \forall x(X^{n-1} x \leftrightarrow \varphi(x))$$

where $\varphi(x)$ *contains neither* X *nor bounded second-order variables of degree n+1*

$$\varepsilon X^{n-2} = \varepsilon Y^{n-1} \leftrightarrow \forall z(X^{n-2} z \leftrightarrow Y^{n-1} z)$$

$$\varepsilon X^{n-1} = \varepsilon Y^{n-1} \leftrightarrow \forall z(X^{n-1} z \leftrightarrow Y^{n-1} z)$$

This predicative hierarchy is also consistent: a proof of the consistency of P^nV for each n = 1, 2, 3 … requires Gentzen arithmetic (Q_4 or $I\Delta_0$(superexp)), and a proof of the consistency of their union $P^{\omega}V$ requires Q_5 or $I\Delta_0$(super^{2}exp). (Burgess 2005, 128-138) Burgess shows that Q is interpretable in PV, and Kalmar arithmetic (Q_3) is interpretable in P^2V. (Burgess 2005, 87-109) On the basis of Burgess' work, Ganea shows that PV is mutually interpretable with Q (Ganea 2007), and Visser shows that P^2V is mutually interpretable with Q_3 (Visser 2009).

However, the above restrictions on comprehension axiom lead to negative results. Although the proof of Hume's Principle from Basic Law V has no use of impredicative comprehension, successor axiom, every number has a successor, can't be proven, relative to Frege's definition, from Hume's Principle within predicative second-order logic; thus Peano Arithmetic can't be proven from the predicative fragments of *Frege Arithmetic*. (Linnebo 2004) Moreover, it is shown that Π^1_1-comprehension is sufficient for proving Peano Arithmetic from Hume's Principle (Heck 2000). Π^1_1-comprehension is:

$$\exists X \forall x (Xx \leftrightarrow \varphi(x)) \quad \textit{where } \varphi(x) \textit{ is a } \Pi^1_1\textit{-formula}$$

Unfortunately, Π^1_1-comprehension is also sufficient for deriving paradox from Basic Law V. Therefore, it seems that Neo-Fregeanism falls into a dilemma: on one hand, impredicative comprehension and Hume's Principle are strong enough to entail Peano Arithmetic, but they both suffer from a downpour of philosophical criticism; on the other hand, predicative comprehension and Basic Law V doesn't entail Peano Arithmetic, though Basic Law V seems more philosophically plausible than Hume's Principle, and predicative comprehension also avoids vicious circle.

At the same time, scholars working on Neo-Fregeanism are moving toward new directions, since there are other theories, in which the strengths of comprehension axioms are between Π^1_0-comprehension and Π^1_1-comprehension. Wehmeier shows that the axiomatic version of Basic Law V is consistent with Δ^1_1-comprehension (Wehmeier 1999). Δ^1_1-comprehension is:

$$\exists X \forall x (Xx \leftrightarrow \varphi(x)) \quad \textit{where } \varphi(x) \textit{ is a } \Delta^1_1\textit{-formula}$$

Furthermore, Ferreira and Wehmeier show that the schematic version of Basic Law V is also consistent with Δ^1_1-comprehension (Ferreira and Wehmeier 2002). However, Walsh shows that successor axiom is still unprovable from Hume's Principle with Δ^1_1-comprehension (Walsh 2009).

In addition to Δ^1_1-comprehension, Burgess shows that if we change Frege's definition of natural number, i.e. we define *Protonatural* as the following:

$$\textit{Protonatural}(x) \leftrightarrow \exists X(\forall y(Xy \leftrightarrow y \lhd x) \land x = \#X \land \neg Xx)$$

where $x \lhd y$ is the abbreviation of the following formula:

$$\exists X \exists Y (x = \#X \land \forall x (Xx \to Yx) \land \exists x (\neg Xx \land Yx) \land y = \#Y)$$

then predicative comprehension and Hume's Principle entail one version of successor axiom (Burgess 2005, 113-117):

$$\forall x (\textit{Protonatural}(x) \to \exists y (\textit{Protonatural}(y) \land Pxy))$$

Moreover, Heck shows if we assign the concept 'natural number' degree two, i.e. we define natural number as the following:

$$\mathrm{N}_2 n \leftrightarrow \forall F_1 (F_1 0 \land \forall x \forall y (F_1 x \land P_1 xy \to F_1 y) \to F_1 n)$$

then ramified predicative comprehension and **Log** entail another version of successor axiom:

$$\forall x (\mathrm{N}_2 x \to \exists y (\mathrm{N}_2 y \land P_1 xy))$$

Log is:

$$\forall x(\varphi(x)\leftrightarrow\psi(x))\rightarrow\#[\lambda x\ \varphi(x)]=\#[\lambda x\ \psi(x)]$$

and Boolos regards **Log** as logical truth. (Heck 2010) However, it is important to keep Frege's definition of natural number, since this definition does capture the ordinary meanings of arithmetic notion (see Heck 2000); thus, changing Frege's definition will not satisfy Neo-Fregeanist ambition. Besides, since a proof of the consistency of P^nV can be formulized in Gentzen arithmetic, the interpretability power of P^nV will not beyond Gentzen arithmetic (according to Gödel's Second Incompleteness Theorem), and there is a long way to go from Gentzen arithmetic to Peano Arithmetic. On the other hand, it has been argued that Heck's ramified theory is very unnatural and complicated (see Linnebo 2009).

Besides, other axioms such as choice axiom and axiom of reducibility can also be added to the predicative fragments of *Grundgesetze*. Ferreira shows that if a finite version of the axiom of reducibility is added to Heck's ramified theory, then the resulting theory is still consistent, and Predicative Peano Arithmetic (ACA_0) is interpretable in it. The axiom of finite reducibility is:

$$\forall H_0\forall F_n(\text{Finite}_0(H_0)\wedge\forall x(F_nx\rightarrow H_0x)\rightarrow\exists G_0\forall x(G_0x\leftrightarrow F_nx))$$

where Finite_0 is defined in terms of "doubly well-ordered". (Ferreira 2005) With the help of reducibility, Ferreira's definition of the concept natural number is one degree less than Heck's definition, and, obviously, Ferreira's theory is stronger than Heck's ramified theory. Moreover, Walsh shows that if Σ^1_1-choice schema is added to the predicative fragment of *Grundgesetze*, the resulting theory is also consistent and interprets the Δ^1_1-fragment of *Grundgesetze*, which means, roughly speaking, the resulting theory is stronger than the Δ^1_1-fragment. Σ^1_1-choice schema is:

$$\forall x\exists X\varphi(x,X)\rightarrow\exists R\forall x\varphi(x,(X)_x)$$

where φ *is* Σ^1_1*-formula, and* $(X)_x=\{y|(y,x)\in R\}$)

Walsh also shows that ACA_0 plus Σ^1_1-choice schema is interpretable in the predicative fragment of *Grundgesetze* plus Σ^1_1-choice schema. (walsh 2009)

Admittedly, the story of Neo-Fregeanism doesn't end here, but Frege's dream of Logicism is still unrealized. Frege was intended to explain that the origin of our knowledge of arithmetic truths is pure logic rather than intuition or perception, and the essential of his Logicism is to logically prove the existence of infinitely many natural numbers. The above failures to prove successor axiom from predicative comprehension seemingly show that the concept 'infinite' is closely related with impredicativity; but impredicativity also easily leads to paradox. Let us reconsider, in Frege's system, the concept, *being the extension of some concept under which it does not fall*. Formally:

$$[\lambda x\ \exists F(x=\varepsilon F\wedge\neg Fx)]$$

If we ask whether its extension falls under it, then paradox arises. This concept includes both a second-order quantifier and a negation. Indeed, this concept doesn't exist if we eliminate any second-order quantifiers from the formulas which define concepts. Also, intuitively, it is possible to avoid paradoxes by making positive restriction on comprehension axiom, i.e. formulas defining concepts are positive. A formula is positive if it is built up from atomic formulas using only conjunction, disjunction, and (first-order and second-order) quantifiers.

2. Consistency and Interpretability

The positive fragment of *Grundgesetze*, consisting of positive comprehension and the schematic version of Basic Law V, is called POV. Its language is:

First-order variables: x, y, z, $\cdots$

Second-order variables: X, Y, Z, $\cdots$ R, S, T, $\cdots$

Logical constants: $\neg$, $\rightarrow$, $\leftrightarrow$, $\wedge$, $\vee$, $\forall$, $\exists$, $\perp$

Identity: $=$

Extension operator: ε

Two most important axioms are the following:

$\exists X \forall x(Xx \leftrightarrow \varphi(x))$ *where* $\varphi(x)$ *doesn't contain negations*

$\varepsilon[\lambda x\ \varphi(x)] = \varepsilon[\lambda x\ \psi(x)] \leftrightarrow \forall x(\varphi(x) \leftrightarrow \psi(x))$

Note that, in comprehension axiom, formulas defining concepts don't contain $\rightarrow$ and $\leftrightarrow$, since these connectives can be defined in terms of negations.

In order to prove the consistency of POV, we need to construct a model satisfying the axioms of POV. Let the first-order variables range over $D=\{0, 1, 2, \cdots\}$, i.e. the set of all natural numbers. Let $(D, \mathfrak{F}_1)$ be topology space, and $\mathfrak{F}_1$ be the co-finite topology on D, i.e. $\mathfrak{F}_1=\{A^c | A$ is finite subset of $D\} \cup \{\emptyset\}$, where A^c is the complement of A. The members of $\mathfrak{F}_1$ are called open sets, and a subset of D is called closed if its complement is open. Let $\mathfrak{B}_1$ be the set of all closed sets of $\mathfrak{F}_1$. Obviously, these closed sets are either D itself or finite subsets of D. In other words, no infinite proper subset of D is contained in $\mathfrak{B}_1$. Specifically, $\mathfrak{B}_1$ contains no sets such as $\{10, 11, 12, 13, \cdots\}$ or $\{2, 4, 6, \cdots\}$. Let the second-order variables range over $\mathfrak{B}_1$. Similarly, let D_n be n-ary Cartesian product over D, and $\mathfrak{B}_n$ be product topology over $(D, \mathfrak{F}_1)$. Then let n-ary relation variables range over $\mathfrak{B}_n$.

A topology space is compact if every open cover has a finite subcover. It can be easily shown that topology space $(D, \mathfrak{F}_1)$ is compact.

We expand the above structure, and add new constants **0**, **1**, **2**$\cdots$. Obviously, the expanded structure satisfies POV if, and only if, the original structure satisfies POV.

Now, we fix denotations for all terms by means of Heck's methods (Heck 1996). As for the new added constant **n**, let it denote natural number 4n+3 in D. As for the value-range terms, their denotations are fixed in the following three steps:

First, fix denotations for all value-range terms containing no second-order variables. The rank of value-range terms is defined as follows: If $\varphi(x)$ contains no value-range terms, then the rank of $\varepsilon[\lambda x\ \varphi(x)]$ is 0; if $\varphi(x)$ contains value-range terms, and the greatest rank of these terms is of n, then the rank of $\varepsilon[\lambda x\ \varphi(x)]$ is n+1. Order these value-range terms in an $\omega\times\omega$ sequence, and make sure that value-range terms of low rank precede those of high rank. Let $L(m,n)$ be a pairing function, and $M(m,n)$ be $2L(m,n)$.

As for the first value-range terms, its denotation is the natural number $M(0, 0)$. Let t be a value-range term. Assume that denotations of all value-range terms preceding t in the sequence of rank are fixed. If $s=\varepsilon[\lambda x\ \beta(x)]$ precedes $t=\varepsilon[\lambda x\ \alpha(x)]$, and $\alpha(x)$ and $\beta(x)$ are equivalent, then assign the denotation of s to t. Otherwise, assign $M(m,n)$ to t, where $M(m,n)$ is the smallest number that has not been assigned. Moreover, the following result also holds: if $\varepsilon[\lambda x\ \varphi(x)]$ and $\varepsilon[\lambda x\ \psi(x)]$ precede t in the sequence of rank, then $\varepsilon[\lambda x\ \varphi(x)]$ and $\varepsilon[\lambda x\ \psi(x)]$ are assigned the same denotation if, and only if, $\varphi(x)$ and $\psi(x)$ are equivalent.

Second, fix denotations for all value-range terms containing free, but no bounded, second-order variables. The above model shows that a set A belongs to $\mathfrak{B}_1$, the domain over which second-order variables range, if, and only if, A is a finite subset of D or D itself. If A is an empty set, then there is a formula $\bot$ such that $A=\varepsilon[\lambda x\ \bot]$. If A is a non-empty finite set, then there is a formula $\varphi(x)$ such that $A=\varepsilon[\lambda x\ \varphi(x)]$, and $\varphi(x)$ is of the form $x=c_1\vee\cdots\vee x=c_n$, where $c_1, \cdots, c_n$ are the new constants added by expansion. If A is D itself, then there is a formula $x=x$ such that $A=\varepsilon[\lambda x\ x=x]$. Therefore, a set A belongs to the domain over which second-order variables range if, and only if, there is a formula $\varphi(x)$ such that $A=\varepsilon[\lambda x\ \varphi(x)]$, where $\varphi(x)$ contains neither second-order variables nor negations. Then we get the following fact: for any interpretation **I** of the free variables in $\varphi(x)$, there is a $\varphi'(x)$, containing no free second-order variables, such that $\varphi'(x)$ and $\varphi(x)$ are equivalent under **I**.

As for a value-range term containing free, but no bounded, second-order variables, let it be $\varepsilon[\lambda x\ \varphi(x)]$, where free second-order variables occurring in $\varphi(x)$ are $X_1,\cdots,X_n$. Fix an interpretation **I**. Assign set $B_i=\varepsilon[\lambda x\ \beta_i(x)]$ to the corresponding X_i, where $\beta_i(x)$ contains no free second-order variables. Replace X_i with $\beta_i(x)$, and let the resulting formula be $\varphi'(x)$. Then $\varphi'(x)$ and $\varphi(x)$ are equivalent under **I**. And we have already fix the denotation of $\varepsilon[\lambda x\ \varphi'(x)]$.

Third, fix denotations for all value-range terms containing bounded, second-order variables. The degree of value-range terms is defined as follows: If $\varepsilon[\lambda x\ \varphi(x)]$ contains no second-order quantifiers, then its degree is 0; if $\varepsilon[\lambda x\ \varphi(x)]$ contains second-order quantifiers, and the greatest degree of value-range terms contained in $\varphi(x)$ is n, then the degree of $\varepsilon[\lambda x\ \varphi(x)]$ is n+1. Order these value-range terms in an $\omega\times\omega$ sequence, and make sure that value-range terms of low degree precede those of high degree. Let $N(m,n)$ be $4L(m,n)+1$.

We have already fix denotations for value-range terms of degree 0. Let t be a value-range term. Assume that denotations of all value-range terms preceding t in the sequence of degree are fixed. If $s=\varepsilon[\lambda x\ \beta(x)]$ precedes $t=\varepsilon[\lambda x\ \alpha(x)]$, and $\alpha(x)$ and $\beta(x)$ are equivalent, then assign the denotation of s to t. Otherwise, assign $N(m,n)$ to t, where $N(m,n)$ is the smallest number that has not been assigned. Moreover, the following result also holds: if $\varepsilon[\lambda x\ \varphi(x)]$ and $\varepsilon[\lambda x\ \psi(x)]$ precede t in the sequence of degree, then $\varepsilon[\lambda x\ \varphi(x)]$ and $\varepsilon[\lambda x\ \psi(x)]$ are assigned the same denotation if, and only if, $\varphi(x)$ and $\psi(x)$ are equivalent.

Now, we show that Basic Law V holds. The above process of fixing denotations for value-range terms shows that $\varepsilon[\lambda x\ \varphi(x)]$ and $\varepsilon[\lambda x\ \psi(x)]$ are assigned the same denotation if, and only if, $\varphi(x)$ and $\psi(x)$ are equivalent, i.e. $\forall x(\varphi(x)\leftrightarrow\psi(x))$.

Then, we show that positive comprehension also holds, i.e. all concepts defined by positive comprehension belong to $\mathfrak{B}_1$. The proof goes in the following steps:

(1) If $\varphi(x)$ is an atomic formula, then $\varphi(x)$ is one of the following forms: $\bot$, $x=x$ or $x=t$, where t is a term. Thus, $\varphi(x)$ correspondingly defines an empty concept, a universal concept or a single concept. If we regard these concepts as sets, then they are all closed sets, i.e. all belong to $\mathfrak{B}_1$.

(2) If $\varphi(x)$ are formulas built up from atomic formulas using only conjunction or disjunction, then $\varphi(x)$ is one of the following form: $x=x\vee\bot$, $x=t\vee\bot$, $x=x\wedge\bot$, $x=t\wedge\bot$, $x=x\vee x=t$, $x=t\vee x=t'$, $x=x\wedge x=t$, $x=t\wedge x=t'$, or more complicated combinations. Thus, $\varphi(x)$ defines a universal concept or a finite concept, which also belongs to $\mathfrak{B}_1$.

The above shows that positive comprehension is permitted to assert the existence of concepts defined by formulas containing no quantifiers. Similarly, it also works for n-ary relations, since they range over the sets of closed sets of product topology.

(3) If $\varphi(x)$ is of the form $\forall y\psi(x,\ y)$, and the concept defined by this formula could be regarded as the set $\{x|\forall y\psi(x,\ y)\}$, then this set can be further regarded as the intersection of all $\{x|\psi(x,\ y)\}$. Concepts defined by formulas containing no quantifiers are closed sets, and arbitrary intersection of closed sets is closed; thus, the concept defined by $\{x|\forall y\psi(x,\ y)\}$ is closed and belongs to $\mathfrak{B}_1$.

(4) $\varphi(x)$ is of the form $\exists y\psi(x,\ y)$. When the existential quantifier is eliminated from $\exists y\psi(x, y)$, the resulting formula $\psi(x, y)$ could defines a binary relation $R\subset D\times D$. Because the topology space D is compact, the projection $j:D\times D\to D$ is a closed map, i.e. taking closed sets to closed sets. Therefore, if the relation defined by $\psi(x, y)$ is closed, then the concept defined by $\exists y\psi(x, y)$ is also closed and belongs to $\mathfrak{B}_1$.

The cases of formulas containing second-order quantifiers are similar to those of first-order, but note that free second-order variables occurring in these formulas could be replaced with first-order formulas.

We complete the proof of the consistency of POV.

Now, we show that Q is interpretable in POV. First, Szmielew-Tarski set theory (ST) is interpretable in POV. The axioms of ST is:

Extensionality $\quad ßx\wedge ßy\wedge\forall z(z\in x\leftrightarrow z\in y)\to x=y$

Null Set $\quad \exists x(ßx\wedge\forall y(y\notin x))$

Adjunction $\quad ßu\to\exists x(ßx\wedge\forall z(z\in x\leftrightarrow z\in u\vee z=v))$

where $ßx$ represents 'x is a set'. $x\in y$ can be defined as $\exists F(y=\varepsilon F\wedge Fx)$. Let variables range over value-range terms. From Basic Law V, we get *Extensionality*. From $\bot$, we get *Null Set*. From $Gx\vee x=b$, we get *Adjunction*. Therefore, ST is interpretable in POV. Second, Burgess has already shown that 'two-sorted theory' is interpretable in ST, and Q is interpretable in 'two-sorted theory' (Burgess 2005, 90-105). The axioms of 'two-sorted theory' are:

(Q1) $\neg x' = 0$

(Q2) $x' = y' \rightarrow x = y$

(R1) $\exists R \forall x \forall y \neg Rxy$

(R2) $\forall R \forall u \forall v \exists S \forall x \forall y (Sxy \leftrightarrow Rxy \vee (x = u \wedge y = v))$

Therefore, by transitivity of interpretability, Q is interpretable in POV.

Although Robinson Arithmetic is very weak, it contains a significant amount of arithmetic; thus, being able to interpret Q shows that POV is not a trivial theory. However, positive comprehension brings no more benefit than predicative comprehension, since PV also interprets Q. At least, it is shown that if we restrict comprehension axiom to '$\varphi(x)$ containing neither second-order quantifiers nor negations', then the resulting theory still interprets Q.

So far, relative to Frege's definitions, Peano Arithmetic is unprovable from POV. Frege's definitions of predecession and natural number both involve negations:

$$Pxy \leftrightarrow \exists F \exists w (Fw \wedge y = \#F \wedge x = \#[\lambda z\ Fz \wedge z \neq w])$$

$$\mathrm{N}n \leftrightarrow \forall F (\forall x (P0x \rightarrow Fx) \wedge \forall x \forall y (Fx \wedge Pxy \rightarrow Fy) \rightarrow Fn) \vee 0 = n$$

In order to prove the most important axiom: successor axiom, we need strong induction:

$$\forall F (F0 \wedge \forall x \forall y (\ \mathrm{N}n \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x (\mathrm{N}x \rightarrow Fx))$$

It is easy to get the following:

$$\forall F (F0 \wedge \forall x \forall y (Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x (\mathrm{N}x \rightarrow Fx))$$

from the definition of natural number, and then we get strong induction by instantiating F with $\mathrm{N}x \wedge Fx$. However, the formula $\mathrm{N}x \wedge Fx$ contains negations; thus, positive comprehension can't define a concept by $\mathrm{N}x \wedge Fx$, let alone instantiating F.

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